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# More comments on superstring interactions in the pp-wave background

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ABSTRACT: We reconsider light-cone superstring field theory on the maximally supersymmetric pp-wave background. We find that the results for the fermionic Neumann matrices given so far in the literature are incomplete and verify our expressions by relating them to the bosonic Neumann matrices and proving several non-trivial consistency conditions among them, as for example the generalization of a flat space factorization theorem for the bosonic Neumann matrices. We also study the bosonic and fermionic constituents of the prefactor and point out a subtlety in the relation between continuum and oscillator basis expressions. We prove that if one requires the vertex to have an exponential form, the use of a different fermionic zero-mode vertex leads to inconsistencies not visible in supergravity.

KEYWORDS: String Field Theory, Penrose limit and pp-wave background.



 http://jhep.sissa.it/archive/papers/jhep092002056/jhep092002056.pdf

## Contents



## 1. Introduction

As is well known by now, besides flat space and $AdS_5 \times S^5$ type-IIB superstring theory admits an additional solution preserving all 32 supersymmetries, the maximally supersymmetric pp-wave background [1]. In contrast to flat space and $AdS_5 \times S^5$ which are related being the asymptotic and near-horizon geometry of the D3-brane, respectively, the maximally supersymmetric pp-wave solution is obtained from $AdS_5 \times S^5$ by the Penrose limit [2], i.e. blowing up the neighborhood of the trajectory of a massless particle rotating around a great circle of the $S^5$. This observation led Berenstein, Maldacena and Nastase (BMN) [3] to a generalization of the AdS/CFT correspondence to the pp-wave background, by proposing that on the Yang-Mills side the Penrose limit is mimicked by focusing on a subset of composite operators of $\mathcal{N} = 4$ SYM with large-R charge. The main importance of this conjectured correspondence lies in the fact that, in contrast to IIB string theory on $AdS_5 \times S^5$ which due to the presence of the RR 5-form flux is still largely intractable, in the light-cone GS formalism type-IIB theory in the maximally supersymmetric pp-wave background is free and thus exactly solvable [4, 5] despite of the RR flux.

Due to the solvability of the theory it is natural to try to include interactions in the picture. In light-cone (closed) superstring field theory interactions are encoded in a cubic interaction vertex, which in flat space was studied in detail by Green, Schwarz and Brink [6, 7] and recently has been generalized to the pp-wave background by Spradlin and Volovich [8, 9]. The interaction vertex can be split into two parts, one — the exponential part of the vertex — contains the Neumann matrices and is necessary to impose the kinematic constraints of the interaction [6, 7]. The second part — the so called prefactor — implements the dynamical constraint that the superalgebra is realized in the interacting theory [6, 7].

In the usual AdS/CFT correspondence it is used that SU($N$) $\mathcal{N}=4$ SYM perturbation theory can be written as a double series expansion in the genus counting parameter $1/N^2$ and the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$. String theory on $AdS_5 \times S^5$ has three parameters, the string length $\sqrt{\alpha'}$, the string coupling $g_s$ and — characterizing the background — the AdS and sphere radius $R$ from which we can form two dimensionless quantities to be matched with corresponding parameters of the gauge theory

$$\frac{R^2}{\alpha'} = \sqrt{\lambda}\,, \qquad \text{and} \qquad 4\pi g_s = g_{\rm YM}^2\,. \tag{1.1}$$

In particular the supergravity approximation, where explicit calculations can be performed, corresponds to strongly coupled gauge theory and this makes a detailed comparison of non-protected quantities very difficult. In the pp-wave case one considers the sector of operators in the SYM theory with U(1) charge $J$ and conformal dimension $\Delta$ such that the difference $\Delta - J$ remains fixed in the limit $J,\, \Delta \to \infty$. One also takes $N \to \infty$ such that $J \sim \sqrt{N}$ and keeps $g_{\rm YM}^2$ finite [3]. There is evidence [10, 11] that in this case the theory can again be expanded in a double series in the effective genus counting parameter $g_2^2 = J^4/N^2$ and the effective coupling $\lambda' = g_{\rm YM}^2 N/J^2$. Moreover the finite quantity $\Delta - J$ turns out to be a function of $\lambda'$ and $g_2^2$ [3, 12, 13]. So we are still left with two parameters on the gauge theory side in the pp-wave limit, although we introduced the quantity $J$ and so $\lambda'$ and $g_2^2$ depend on three parameters. However, since both $J$ and $N$ are strictly taken to infinity, only the ratio $J^2/N$ survives as a true parameter. Light-cone string theory in the pp-wave background effectively has three parameters, the string length $|\alpha| \equiv \alpha'|p^+|$, the string coupling $g_s$ and the scale $\mu$ characterizing the background. The matching in this case is [3]

$$\frac{1}{(\mu\alpha)^2} = \lambda'\,, \qquad \text{and} \qquad 4\pi g_s(\mu\alpha)^2 = g_2\,. \tag{1.2}$$

An interesting property of the duality between string theory on the pp-wave and the subsector of composite operators in $\mathcal{N}=4$ SYM with large-R charge is that in this case both sides of the duality are simultaneously accessible in their perturbative regimes.

A definite proposal which quantity on the field theory side should be matched with three-string amplitudes was put forward in [11] for the free, planar limit corresponding to $\mu = \infty$ [1] on the string theory side. Various checks of this proposal and further studies of cubic interactions in the pp-wave background and their comparison to field theory were done in [9] and [14]–[18]. So called vector operators were studied in [19], whose results suggest that the proposal of [11] has to be modified for mixed scalar/vector operators, in order to match string theory predictions.[2] An alternative resolution — to use a different fermionic zero-mode vertex as compared to [8] — was proposed in [20]. However, in appendix C

[1]Whenever we write $\mu = \infty$ or $\mu \to 0$ we actually mean the dimensionless quantity $\mu|\alpha|$.

[2]I would like to thank Marcus Spradlin for emphasizing this point to me.



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of this paper we prove that this choice is not consistent, if one requires the vertex to have an exponential form. The structure of the interaction vertex for large $\mu$, in particular subleading corrections which are important in trying to generalize the proposal of [11] to the interacting gauge theory were studied in [21, 22]. The latter has some overlap with results presented in section 4 of this paper. Additional work on string interactions can be found in [23, 24], an interesting new development concerning string theory on non-maximally supersymmetric plane wave backgrounds is [25]. For the covariant formulation of string theory on plane wave backgrounds see [26, 27]. A first study of four-point correlation functions in the BMN limit was initiated in [28].

This paper is organized as follows. We begin with a brief review of the free light-cone string theory in section 2. In section 3 we discuss the interacting theory. The first part of this section deals with the bosonic contribution to the exponential part of the vertex and is included for completeness, all of the presented formulae were already derived in [8]. The second part of section 3 deals with the fermionic contribution to the exponential part of the vertex and slightly differs from the original result of [8]. A detailed proof of the expressions presented there is relegated to appendix B. Section 4 deals with properties of the Neumann matrices and the flat space limit of the exponential part of the vertex, in particular we generalize a flat space factorization theorem [6] to the pp-wave background, see also [22] were the same result is obtained. In section 5 we study in some detail the constituents of the prefactor. We begin with the oscillator expressions of the bosonic part which were already obtained in [9]. Using results of section 4 we then verify explicitly that the operator expressions [6, 7] for the bosonic constituents coincide up to a possibly $\mu$-dependent normalization with the oscillator expressions when acting on the vertex. In the second part of section 5 we consider the fermionic constituent and relate the vector appearing in the oscillator expression to its bosonic counterpart and check that it has the correct flat space limit of [7]. In the end of this section we prove that in the fermionic case a non-trivial $\mu$-dependent matrix appears in the relation between operator and oscillator expressions. We finish with conclusions in section 6, summarize some known identities [6] in appendix A and present the proof dealing with the fermionic zero-mode part of the vertex in appendix C.

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## 2. Review of free string theory on the pp-wave

In this section we review some known facts about free light-cone string field theory in the maximally supersymmetric pp-wave background [4, 5, 8]. We begin with the bosonic part of the string light-cone action in the pp-wave background [4, 5]

$$S_{\mathrm{l.c.}} = \frac{e(\alpha)}{4\pi\alpha'} \int d\tau \int_0^{2\pi|\alpha|} d\sigma \left[\dot{x}^2 - \acute{x}^2 - \mu^2 x^2\right] , \tag{2.1}$$

where $\dot{x} = \partial_\tau x$, $\acute{x} = \partial_\sigma x$, $|\alpha| \equiv \alpha'|p^+|$, $e(\alpha) \equiv \mathrm{sign}(\alpha)$ and $p^+ < 0$ ($p^+ > 0$) for incoming (outgoing) strings. We suppress the transverse index. The mode expansions of the fields

$x(\sigma,\tau)$ and $p(\sigma,\tau)$ at $\tau=0$ are

$$
\begin{aligned}
x(\sigma) &= x_0 + \sqrt{2}\sum_{n=1}^{\infty}\left(x_n\cos\frac{n\sigma}{|\alpha|} + x_{-n}\sin\frac{n\sigma}{|\alpha|}\right), \\
p(\sigma) &= \frac{1}{2\pi|\alpha|}\left[p_0 + \sqrt{2}\sum_{n=1}^{\infty}\left(p_n\cos\frac{n\sigma}{|\alpha|} + p_{-n}\sin\frac{n\sigma}{|\alpha|}\right)\right],
\end{aligned}
\tag{2.2}
$$

where in terms of oscillators

$$
x_n = i\sqrt{\frac{\alpha'}{2\omega_n}}\left(a_n - a_n^\dagger\right), \qquad p_n = \sqrt{\frac{\omega_n}{2\alpha'}}\left(a_n + a_n^\dagger\right), \qquad \left[a_n, a_m^\dagger\right] = \delta_{nm} \tag{2.3}
$$

and $\omega_n = \sqrt{n^2 + (\mu\alpha)^2}$. The light-cone hamiltonian is

$$
H = \frac{1}{\alpha}\sum_{n\in\mathbb{Z}}\omega_n\left(a_n^\dagger a_n + 4\right). \tag{2.4}
$$

The zero-point energy will be cancelled by the fermionic contribution. The hamiltonian only depends on the two dimensionful quantities $\mu$ and $\alpha$, i.e. $\alpha'$ and $p^+$ should not be treated separately.

The fermionic part of the action is [4, 5]

$$
S_{\text{l.c.}} = \frac{1}{8\pi}\int d\tau \int_0^{2\pi|\alpha|} d\sigma\left[i(\bar\vartheta\dot\vartheta + \vartheta\dot{\bar\vartheta}) - \vartheta\acute{\vartheta} + \bar\vartheta\grave{\bar\vartheta} - 2\mu\bar\vartheta\Pi\vartheta\right], \tag{2.5}
$$

where $\vartheta^a$ is a complex positive chirality spinor of SO(8) (we mostly suppress the index $a$) and $\Pi = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$ is symmetric and traceless, $\Pi^2 = 1$.

The mode expansion of $\vartheta$ and its conjugate momentum $i\lambda \equiv \frac{i}{4\pi}\bar\vartheta$ at $\tau = 0$ is

$$
\begin{aligned}
\vartheta(\sigma) &= \vartheta_0 + \sqrt{2}\sum_{n=1}^{\infty}\left(\vartheta_n\cos\frac{n\sigma}{|\alpha|} + \vartheta_{-n}\sin\frac{n\sigma}{|\alpha|}\right), \\
\lambda(\sigma) &= \frac{1}{2\pi|\alpha|}\left[\lambda_0 + \sqrt{2}\sum_{n=1}^{\infty}\left(\lambda_n\cos\frac{n\sigma}{|\alpha|} + \lambda_{-n}\sin\frac{n\sigma}{|\alpha|}\right)\right]
\end{aligned}
\tag{2.6}
$$

with the reality condition $\lambda_n = \frac{|\alpha|}{2}\bar\vartheta_n$. The anticommutation relation $\{\vartheta^a(\sigma), \lambda^b(\sigma')\} = \delta^{ab}\delta(\sigma - \sigma')$ follows from $\{\vartheta_n^a, \lambda_m^b\} = \delta^{ab}\delta_{n,m}$. To write the hamiltonian in canonical form in terms of fermionic operators $b_n$ satisfying $\{b_n, b_m^\dagger\} = \delta_{nm}$ define [8]

$$
\vartheta_n = \frac{c_n}{\sqrt{|\alpha|}}\left[(1 + \rho_n\Pi)b_n + e(\alpha n)(1 - \rho_n\Pi)b_{-n}^\dagger\right], \tag{2.7}
$$

where $e(n) = \text{sign}(n)$, $(e(0) \equiv 1)$ and

$$
\rho_n = \rho_{-n} = \frac{\omega_n - |n|}{\mu\alpha}, \qquad c_n = c_{-n} = \frac{1}{\sqrt{1 + \rho_n^2}}. \tag{2.8}
$$

Then the fermionic part of the light-cone hamiltonian is

$$
H = \frac{1}{\alpha}\sum_{n\in\mathbb{Z}}\omega_n\left(b_n^\dagger b_n - 4\right). \tag{2.9}
$$

Notice that in the limit $\mu \to 0$ we have $c_n(1 \pm \rho_n \Pi) \to 1$ and therefore

$$\lim_{\mu\to 0} \vartheta_n = \frac{1}{\sqrt{|\alpha|}} \left( b_n + e(\alpha n) b^\dagger_{-n} \right) . \tag{2.10}$$

This allows to relate the $b_n$ to the $Q_n^{\mathrm{I,II}}$ of [7] which will be useful when checking that the fermionic vertex has the correct flat space limit. For $n > 0$ we have in the limit $\mu \to 0$

$$b_n \longleftrightarrow \frac{e(\alpha)}{\sqrt{|\alpha|}} Q_n^{\mathrm{I}} , \qquad b_{-n} \longleftrightarrow \frac{e(\alpha)}{\sqrt{|\alpha|}} Q_n^{\mathrm{II}} , \tag{2.11}$$

and $[Q_n^{\mathrm{I,II}}]^\dagger = e(\alpha) Q_{-n}^{\mathrm{I,II}}$.

## 3. Interacting string theory

As already stated in the introduction, the cubic interaction vertex of light-cone string field theory can be split into two parts, determined by imposing the kinematic and dynamic constraints of the interaction. In this section we study the exponential part of the vertex which deals with the kinematic constraints. The prefactor that is determined by the dynamic constraints is the subject of section 5. The bosonic contribution to the exponential part of the three-string interaction vertex has to satisfy the kinematic constraints [6, 7]

$$\sum_{r=1}^{3} p_r(\sigma_r)|E_a\rangle = 0 , \qquad \sum_{r=1}^{3} e(\alpha_r) x_r(\sigma_r)|E_a\rangle = 0 . \tag{3.1}$$

It can be obtained by evaluating the integral [6, 7]

$$\begin{aligned} |E_a\rangle &= \prod_{r=1}^{3} \int dp_r \psi(p_r) \Delta^8 \left[ \sum_{s=1}^{3} p_s(\sigma_s) \right] |0\rangle \\ &= \prod_{r=1}^{3} \prod_{n\in\mathbb{Z}} \int dp_{n(r)} \psi(p_{n(r)}) \delta^8 \left[ \sum_{s=1}^{3} \left( X^{(s)} p_s \right)_n \right] |0\rangle . \end{aligned} \tag{3.2}$$

$\Delta^8[\sum_{s=1}^3 p_s(\sigma_s)]$ is the Delta-functional guaranteeing the continuous overlap of the string worldsheets in the interaction and $\psi(p_{n(r)})$ is the harmonic oscillator wavefunction for occupation number $n$

$$\psi(p_{n(r)}) = \left( \frac{\omega_{n(r)}\pi}{\alpha'} \right)^{-1/4} \exp\left( -\frac{\alpha'}{(2\omega_{n(r)})} p_{n(r)}^2 + \sqrt{\frac{2\alpha'}{\omega_{n(r)}}} a^\dagger_{n(r)} p_{n(r)} - \frac{1}{2} a^\dagger_{n(r)} a^\dagger_{n(r)} \right) . \tag{3.3}$$

The coordinates of the three strings are parameterized by

$$\begin{aligned} \sigma_1 &= \sigma , \qquad -\pi\alpha_1 \le \sigma \le \pi\alpha_1 , \\ \sigma_2 &= \begin{cases} \sigma - \pi\alpha_1 , & \pi\alpha_1 \le \sigma \le \pi(\alpha_1+\alpha_2) , \\ \sigma + \pi\alpha_1 , & -\pi(\alpha_1+\alpha_2) \le \sigma \le -\pi\alpha_1 , \end{cases} \\ \sigma_3 &= -\sigma , \qquad -\pi(\alpha_1+\alpha_2) \le \sigma \le \pi(\alpha_1+\alpha_2) . \end{aligned} \tag{3.4}$$

Here $\alpha_1+\alpha_2+\alpha_3=0$ and $\alpha_3<0$. We also use $\beta\equiv\alpha_1/\alpha_3$ and $\beta+1=-\alpha_2/\alpha_3$. The full Delta-functional takes the form [6, 7]

$$\Delta\left[\sum_{r=1}^{3} p_r(\sigma_r)\right]\sim\prod_{m\in\mathbb{Z}}\delta\left(\sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} X_{mn}^{(r)} p_{n(r)}\right). \tag{3.5}$$

We ignored factors of $\sqrt{2}$ which can be absorbed in the measure. The matrices $X_{mn}^{(r)}$ have the following structure [6, 7][3]

$$X_{mn}^{(r)}\equiv\begin{cases} X_{mn}^{(r)}\,, & m>0\,,n>0\,,\\ \frac{\alpha_3}{\alpha_r}\frac{n}{m}X_{-m,-n}^{(r)}\,, & m<0\,,n<0\,,\\ \frac{1}{\sqrt{2}}X_{m0}^{(r)}\,, & m>0\,,\\ 1\,, & m=0=n\,,\\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.6}$$

Here [6, 7]

$$X_{mn}^{(1)}\equiv(-1)^n\frac{2m\beta}{\pi}\frac{\sin m\pi\beta}{m^2\beta^2-n^2}\,,\qquad X_{mn}^{(2)}\equiv\frac{2m(\beta+1)}{\pi}\frac{\sin m\pi\beta}{m^2(\beta+1)^2-n^2} \tag{3.7}$$

and $X_{mn}^{(3)}=\delta_{mn}$. It is standard to perform the gaussian integral (3.2) and the result is [6, 7, 8]

$$|E_a\rangle\sim\exp\left(\frac{1}{2}a_r^{\dagger T}\overline{N}^{rs}a_s^\dagger\right)|0\rangle\,, \tag{3.8}$$

where $r,s\in\{1,2,3\}$. The determinant factor coming from the functional determinants will be cancelled by the fermionic contribution except for the zero-mode part which is proportional to $(\frac{\mu}{\alpha'})^2(\frac{\alpha_1\alpha_2}{\alpha_3})^2$.

The Neumann matrices are [8]

$$\overline{N}_{mn}^{rs}=\delta^{rs}\delta_{mn}-2\left(C_{(r)}^{1/2}X^{(r)T}\Gamma_a^{-1}X^{(s)}C_{(s)}^{1/2}\right)_{mn}\,, \tag{3.9}$$

where

$$\left[C_{(r)}\right]_{mn}=\omega_{m(r)}\delta_{mn}\,,\qquad\text{and}\qquad\Gamma_a=\sum_{r=1}^{3}X^{(r)}C_{(r)}X^{(r)T}\,. \tag{3.10}$$

From the structure of the $X^{(r)}$ it follows that $\Gamma_a$ is block diagonal and using the identities (A.5) in appendix A one can write the blocks as [8] ($C_{mn}=m\delta_{mn}$)

$$[\Gamma_a]_{mn}=\begin{cases}\left(C^{1/2}\Gamma C^{1/2}\right)_{mn}\,, & m,n>0\,,\\ -2\mu\alpha_3\,, & m=0=n\,,\\ \left(C^{1/2}\Gamma_- C^{1/2}\right)_{-m,-n}\,, & m,n<0\,,\end{cases} \tag{3.11}$$

where

$$\Gamma\equiv\sum_{r=1}^{3}A^{(r)}U_{(r)}A^{(r)T}\,,\qquad\Gamma_-\equiv\sum_{r=1}^{3}A_-^{(r)}U_{(r)}^{-1}A_-^{(r)T}\,. \tag{3.12}$$

[3]The matrices $X_{mn}^{(r)}$ actually differ from the ones of [6, 7] by a factor of $(-1)^m$, which however has no physical significance.

Here $A^{(r)}_{mn} = [C^{-1/2} X^{(r)} C^{1/2}]_{mn}$ for $m,\, n > 0$ and

$$\begin{aligned} U_{(r)} &= C^{-1}\left(C_{(r)} - \mu\alpha_r \mathbf{1}\right), \\ U^{-1}_{(r)} &= C^{-1}\left(C_{(r)} + \mu\alpha_r \mathbf{1}\right), \qquad A^{(r)}_{-} = \frac{\alpha_3}{\alpha_r} C^{-1} A^{(r)} C\,. \end{aligned} \tag{3.13}$$

The matrix $\Gamma$ (which reduces to the flat space $\Gamma$ of [6, 7] for $\mu \to 0$) exists and is invertible, whereas $\Gamma_-$ is ill-defined since the above sum is divergent. Nevertheless it is possible to derive a well-defined identity for $\Gamma_-^{-1}$ [8]

$$\Gamma_-^{-1} = U_{(3)}\left(1 - \Gamma^{-1} U_{(3)}\right). \tag{3.14}$$

Since $\Gamma_-^{-1}$ is related to $\Gamma^{-1}$ it is possible to relate the Neumann matrices with positive and negative indices. The only nonvanishing matrix elements with negative indices are $\overline{N}^{rs}_{-m,-n}$ for $m,\, n > 0$ related to $\overline{N}^{rs}_{mn}$ via [8]

$$\overline{N}^{rs}_{-m,-n} = -\left(U_{(r)} \overline{N}^{rs} U_{(s)}\right)_{mn}. \tag{3.15}$$

Analogously to the bosonic contribution, the fermionic exponential part of the interaction vertex has to satisfy [6, 7]

$$\sum_{r=1}^{3} \lambda_{(r)}(\sigma_r)|E_b\rangle = 0\,, \qquad \sum_{r=1}^{3} e(\alpha_r)\vartheta_{(r)}(\sigma_r)|E_b\rangle = 0\,. \tag{3.16}$$

As in the bosonic case it could be obtained by constructing the fermionic analogue of the wavefunction (3.3) and then performing the resulting integrals over the non-zero-modes. The pure zero-mode contribution has to be treated separately. Notice that due to the structure of the $X^{(r)}_{mn}$ the exponential will — as in flat space [6, 7] — contain a part which is linear in zero-mode oscillators. Instead of directly performing the functional integral the exponential can be obtained (up to the normalization) by making a suitable ansatz and imposing the constraints (3.16) [6, 7]. We find the following expression (cf. appendix B for the details; the notation is defined below)

$$|E_b\rangle \sim \exp\left[\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty} b^\dagger_{-m(r)} Q^{rs}_{mn} b^\dagger_{n(s)} - \sqrt{2}\Lambda \sum_{r=1}^{3}\sum_{m=1}^{\infty} Q^r_m b^\dagger_{-m(r)}\right] |E^0_b\rangle\,, \tag{3.17}$$

where $|E^0_b\rangle$ is the pure zero-mode part of the fermionic vertex (see also the discussion below and appendix C)

$$|E^0_b\rangle = \prod_{a=1}^{8}\left[\sum_{r=1}^{3} \lambda^a_{0(r)}\right]|0\rangle \tag{3.18}$$

and manifestly satisfies $\sum_{r=1}^3 \lambda_{0(r)}|E^0_b\rangle = 0$ and $\sum_{r=1}^3 \alpha_r \vartheta_{0(r)}|E^0_b\rangle = 0$. Notice that $|0\rangle$ is not the vacuum defined to be annihilated by the $b_{0(r)}$. Rather it satisfies $\vartheta_{0(r)}|0\rangle = 0$ and



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$H_{(r)}|0\rangle = 4\mu e(\alpha_r)|0\rangle$ so that the zero should be thought of as the occupation number. In the limit $\mu \to 0$ it coincides with the usual flat space vacuum. Furthermore

$$\begin{aligned}
Q^{rs}_{mn} &= e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}\left[P^{-1}_{(r)}U_{(r)}C^{1/2}\overline{N}^{rs}C^{-1/2}U_{(s)}P^{-1}_{(s)}\right]_{mn}, \\
Q^r_n &= \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\left[P_{(r)}C^{1/2}_{(r)}C^{1/2}\overline{N}^r\right]_n, \\
\Lambda &= \alpha_1\lambda_{0(2)} - \alpha_2\lambda_{0(1)}\,.
\end{aligned} \tag{3.19}$$

Here we introduced some more notation, namely $\alpha \equiv \alpha_1\alpha_2\alpha_3$,

$$P_{n(r)} \equiv \frac{1-\rho_{n(r)}\Pi}{\sqrt{1-\rho^2_{n(r)}}}, \qquad K \equiv -\frac{1}{4}B^T\Gamma^{-1}B\,, \qquad \overline{N}^r \equiv -C^{-1/2}A^{(r)T}\Gamma^{-1}B \tag{3.20}$$

and the vector $B_m$ is related to $X^{(r)}_{m0}$ via $X^{(r)}_{m0} = -\varepsilon^{rs}\alpha_s(C^{1/2}B)_m$. The scalar $K$ and the vector $\overline{N}^r$ reduce to the quantities defined in [6, 7] in the flat space limit. Notice that our result (3.17) is slightly different from the one obtained in [8], in particular the terms proportional to $\Lambda$ were absent. We will show in section 4 that as $\mu \to 0$ our expression (3.17) coincides with the flat space result of [7]. This however does not really prove that the above expression is correct since some structure is lost in the flat space limit. The analysis of appendix B however uniquely fixes the above expressions. Another important point is the following: our computations in appendix B show that it is essential for the consistency of the interaction that the zero-mode part $|E^0_b\rangle$ is annihilated by the combination $\vartheta_{0(1)} - \vartheta_{0(2)}$ (as in flat space). This is a constraint induced by the non-zero-modes and hence not visible in supergravity. In a recent paper [20] it was proposed that in order to resolve present discrepancies [19] with the field theory proposal of [11] one should use a different zero-mode vertex $|\delta\rangle$ built on the vacuum annihilated by all the $b_{0(r)}$. We give a proof in appendix C, that this choice is not consistent if one requires the vertex to have an exponential form. In particular it should also satisfy $(\vartheta_{0(1)} - \vartheta_{0(2)})|\delta\rangle = 0$, which is not the case.

## 4. Properties of the Neumann matrices and the flat space limit

In this section we analyze in more detail the exponential part of the vertex presented in the previous section. We check that in the flat space limit all the expressions reduce to the known ones derived long ago by Green, Schwarz and Brink [7]. As already stated this is only a consistency check which nevertheless is illuminating. A detailed proof of the expressions appearing in the exponential part of the vertex is presented in appendix B. Furthermore we generalize a flat space factorization theorem [6] for the bosonic Neumann matrices to the pp-wave background (the same expression was obtained independently in [22]) which might be useful for various purposes, such as the comparison of string and field theory computations.



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For $m,n>0$ the nonvanishing elements of the bosonic Neumann matrices are [8, 9]

$$\begin{aligned}
\overline{N}^{rs}_{mn} &= \delta^{rs}\delta_{mn} - 2\sqrt{\frac{\omega_{m(r)}\omega_{n(s)}}{mn}}\left(A^{(r)\,T}\Gamma^{-1}A^{(s)}\right)_{mn}\,, && (4.1)\\
\overline{N}^{rs}_{m0} &= -\sqrt{2\mu\alpha_s\omega_{m(r)}}\varepsilon^{st}\alpha_t\overline{N}^{r}_{m}\,, \qquad s\in\{1,2\}\,, && (4.2)\\
\overline{N}^{rs}_{00} &= \left(\delta^{rs}+\frac{\sqrt{\alpha_r\alpha_s}}{\alpha_3}\right)(1-4\mu\alpha K)\ , \qquad r,s\in\{1,2\}\,, && (4.3)\\
\overline{N}^{r3}_{00} &= \delta^{r3}-\sqrt{\left|\frac{\alpha_r}{\alpha_3}\right|}, \qquad r\in\{1,2,3\}\,. && (4.4)
\end{aligned}$$

The terms in $\overline{N}^{rs}_{00}$ and $\overline{N}^{r3}_{00}$ that are not proportional to $\mu$ give the pure supergravity contribution to the Neumann matrices. The part of $\overline{N}^{rs}_{00}$ that is proportional to $\mu$ is induced by positive string modes of $p_3$. The only nonvanishing matrix elements with negative indices are $\overline{N}^{rs}_{-m,-n}$ related to the Neumann matrices with positive indices by (3.15). In flat space $\overline{N}^{rs}_{mn}$ is related to $\overline{N}^{r}_{m}\overline{N}^{s}_{n}$ via[4] [6]

$$\overline{N}^{rs}_{mn} = -\alpha\frac{(mn)^{3/2}}{\alpha_r n+\alpha_s m}\overline{N}^{r}_{m}\overline{N}^{s}_{n}\,. \qquad (4.5)$$

Below we will derive a generalization of this formula for all $\mu$ (see also [22]). Let us introduce $\Upsilon\equiv\sum_{r=1}^{3}A^{(r)}U^{-1}_{(r)}A^{(r)\,T}=\Gamma+\mu\alpha BB^T$ (cf. (A.5)). Its inverse is related to $\Gamma^{-1}$ by (see also [22])

$$\Upsilon^{-1}=\Gamma^{-1}-\frac{\mu\alpha}{1-4\mu\alpha K}\left(\Gamma^{-1}B\right)\left(\Gamma^{-1}B\right)^T \qquad (4.6)$$

and thus

$$\Upsilon^{-1}B=\frac{1}{1-4\mu\alpha K}\Gamma^{-1}B\,. \qquad (4.7)$$

Then one can show that the following relations hold

$$\begin{aligned}
A^{(r)\,T}C^{-1}U_{(3)}\Gamma^{-1} &= A^{(r)\,T}C^{-1}+\frac{\alpha_r}{\alpha_3}C^{-1}U_{(r)}A^{(r)\,T}\Gamma^{-1}\,, \qquad r\in\{1,2\}\,, && (4.8)\\
\Upsilon^{-1}U^{-1}_{(3)}C^{-1}A^{(r)} &= C^{-1}A^{(r)}+\frac{\alpha_r}{\alpha_3}\Upsilon^{-1}A^{(r)}U^{-1}_{(r)}C^{-1}\,, \qquad r\in\{1,2\}\,, && (4.9)\\
2C^{-1} &= \Gamma^{-1}U_{(3)}C^{-1}+C^{-1}U_{(3)}\Gamma^{-1}+\Upsilon^{-1}U^{-1}_{(3)}C^{-1}+\\
&\quad +C^{-1}U^{-1}_{(3)}\Upsilon^{-1}-\alpha_1\alpha_2\Upsilon^{-1}B\left(\Gamma^{-1}B\right)^T\,. && (4.10)
\end{aligned}$$

From here we find using (4.6), (4.7) and (A.4)

$$\begin{aligned}
\overline{N}^{rs}_{mn} &= -(1-4\mu\alpha K)^{-1}\frac{\alpha}{\alpha_r\omega_{n(s)}+\alpha_s\omega_{m(r)}}\times\\
&\quad\times\left[U^{-1}_{(r)}C^{1/2}_{(r)}C\overline{N}^{r}\right]_m\left[U^{-1}_{(s)}C^{1/2}_{(s)}C\overline{N}^{s}\right]_n\,. && (4.11)
\end{aligned}$$

Clearly eq. (4.11) reduces to eq. (4.5) as $\mu\to 0$. It coincides with the result obtained by [22]. It is useful to write the above in matrix form as

$$\alpha_s C_{(r)}\overline{N}^{rs}+\alpha_r\overline{N}^{rs}C_{(s)}=-\alpha(1-4\mu\alpha K)^{-1}U^{-1}_{(r)}C^{1/2}_{(r)}C\overline{N}^{r}\left[U^{-1}_{(s)}C^{1/2}_{(s)}C\overline{N}^{s}\right]^T\,. \qquad (4.12)$$

[4]Notice that in comparison with [6] we have $\overline{N}^{rs}_{\rm here}=C^{1/2}\overline{N}^{rs}_{\rm there}C^{1/2}$.

Using the above we can also give a simple expression for $Q^{rs}_{mn}$

$$\begin{aligned} Q^{rs}_{mn} &= -e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|\frac{m}{n}}(1-4\mu\alpha K)^{-1}\frac{\alpha}{\alpha_r\omega_{n(s)}+\alpha_s\omega_{m(r)}}\times \\ &\quad\times\left[P^{-1}_{(r)}C^{1/2}_{(r)}C\overline{N}^{r}\right]_m\left[P^{-1}_{(s)}C^{1/2}_{(s)}C\overline{N}^{s}\right]_n . \end{aligned} \tag{4.13}$$

In what follows we will show that as $\mu\to 0$ the expressions for $|E_a\rangle$ and $|E_b\rangle$ (cf. (3.8) and (3.17)) coincide with the flat space expressions of [7]. We begin with the bosonic contribution. In the limit $\mu\to 0$ and for $n>0$ the $a_n$ are related to the $a_n^{\mathrm{I,II}}$ of [7] as

$$a_n \longleftrightarrow \frac{1}{\sqrt{n}}a_n^{\mathrm{I}}\,,\qquad a_{-n}\longleftrightarrow\frac{i}{\sqrt{n}}a_n^{\mathrm{II}} \tag{4.14}$$

and $[a_n^{\mathrm{I,II}}]^\dagger = a_{-n}^{\mathrm{I,II}}$. Rewrite

$$\begin{aligned} \frac{1}{2}\sum_{r,s}\sum_{m,n\in\mathbb{Z}} a^\dagger_{m(r)}\overline{N}^{rs}_{mn}a^\dagger_{n(s)} &= \frac{1}{2}\sum_{r,s}\left[\sum_{m,n=1}^{\infty}a^\dagger_{m(r)}\overline{N}^{rs}_{mn}a^\dagger_{n(s)}+\sum_{m,n=1}^{\infty}a^\dagger_{-m(r)}\overline{N}^{rs}_{-m,-n}a^\dagger_{-n(s)}+\right. \\ &\quad\left.+\;2\sum_{m=1}^{\infty}a^\dagger_{m(r)}\overline{N}^{rs}_{m0}a^\dagger_{0(s)}+a^\dagger_{0(r)}\overline{N}^{rs}_{00}a^\dagger_{0(s)}\right]. \end{aligned} \tag{4.15}$$

Consider first the part linear in non-zero-mode oscillators. We have

$$\begin{aligned} &\sum_{r,s=1}^{3}\sum_{m=1}^{\infty}a^\dagger_{m(r)}\overline{N}^{rs}_{m0}a^\dagger_{0(s)} = \\ &\quad= \sqrt{2\mu}\sum_{r=1}^{3}\sum_{m=1}^{\infty}\sqrt{\omega_{m(r)}}a^\dagger_{m(r)}\overline{N}^{r}_{m}\left(\alpha_1\sqrt{\alpha_2}a^\dagger_{0(2)}-\alpha_2\sqrt{\alpha_1}a^\dagger_{0(1)}\right). \end{aligned} \tag{4.16}$$

Using

$$a_0^\dagger = \sqrt{\frac{\alpha'}{2\mu|\alpha|}}\,p_0 + i\sqrt{\frac{\mu|\alpha|}{2\alpha'}}\,x_0 \tag{4.17}$$

this can be further written as

$$\sqrt{\alpha'}\sum_{r=1}^{3}\sum_{m=1}^{\infty}\sqrt{\omega_{m(r)}}a^\dagger_{m(r)}\overline{N}^{r}_{m}\left(\mathbb{P}-i\frac{\mu\alpha}{\alpha'}\mathbb{R}\right). \tag{4.18}$$

Here $\mathbb{P}=\alpha_1 p_{0(2)}-\alpha_2 p_{0(1)}$ and $\alpha_3\mathbb{R}=x_{0(1)}-x_{0(2)}$, $[\mathbb{R},\mathbb{P}]=i$. The above expression reduces to the one in flat space [7] for $\mu\to 0$. The part quadratic in non-zero-mode oscillators obviously has the correct flat space limit using (3.15).

Finally the part quadratic in zero-mode oscillators is

$$\sum_{r,s=1}^{3}a^\dagger_{0(r)}\overline{N}^{rs}_{00}a^\dagger_{0(s)} = \sum_{r,s=1}^{3}a^\dagger_{0(r)}M^{rs}_{\mathrm{Sugra}}a^\dagger_{0(s)}+2K\alpha'\left(\mathbb{P}-i\frac{\mu\alpha}{\alpha'}\mathbb{R}\right)^2. \tag{4.19}$$

In the limit $\mu\to 0$ the second term reduces to the result in flat space [7]. The first term is divergent in the limit which is due to the fact that the supergravity particles are no longer confined by the harmonic oscillator potential and propagate in infinite volume.



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Now consider the flat space limit of the fermionic expression $|E_b\rangle$ in (3.17). We will need that $\lim_{\mu\to 0} P^{-1}_{n(r)} = 1$. Then we see from eqs. (3.19) that

$$\lim_{\mu\to 0} Q^{rs}_{mn} = \frac{\sqrt{|\alpha_r\alpha_s|}}{\alpha_r}\left[C^{1/2}\overline{N}^{rs}C^{-1/2}\right]_{mn}, \qquad \lim_{\mu\to 0} Q^{r}_{m} = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\left[C\overline{N}^{r}\right]_{m}. \tag{4.20}$$

Taking into account (2.11) we see that $|E_b\rangle$ precisely coincides with the flat space expression eq. (4.21) of [7].

## 5. The constituents of the prefactor

The full expression for the superstring vertex involves an operator $G$, the so-called prefactor, necessary to ensure the realization of the superalgebra in the interacting theory [6]. In flat space this operator, when written in the continuum basis depends on $p_r(\sigma)$, $\acute{x}_r(\sigma)$ and $\lambda_r(\sigma)$. Since $p_r(\sigma)$ and $\lambda_r(\sigma)$ correspond to functional derivatives with respect to $x_r(\sigma)$ and $\vartheta_r(\sigma)$ the only physically sensible value of $\sigma$ to choose is the interaction point $\sigma = \pm\pi\alpha_1$. Since operators at this point are singular the prefactor must be carefully defined in the limit $\sigma \to |\pi\alpha_1|$ [6]. In the end one obtains an expression containing both creation and annihilation operators of the various oscillators. The annihilation operators can be eliminated by (anti)commuting them through the exponential factors of the vertex. We then write $GE_aE_b|0\rangle = E_aE_b\widetilde{G}|0\rangle$, where $\widetilde{G}$ only contains the creation operators.

In this section we study the bosonic and fermionic constituents of the prefactor, relate the infinite component vectors that appear in their oscillator basis expressions and point out a subtlety in the relation between the continuum and oscillator basis expressions which is not present in the flat space case. This might have the consequence that the precise form of the prefactor derived in [8] has to be changed.

### 5.1 The bosonic constituents

An important constraint on the prefactor is that it must respect the local conservation laws ensured by $|E_a\rangle$ and $|E_b\rangle$. For the bosonic part this means that it must commute with [6, 7]

$$\left[\sum_{r=1}^{3} p_r(\sigma), \widetilde{G}\right] = 0 = \left[\sum_{r=1}^{3} e(\alpha_r)x_r(\sigma), \widetilde{G}\right]. \tag{5.1}$$

Consider first an expression of the form

$$\mathcal{K}_0 + \mathcal{K}_+ = \sum_{r=1}^{3}\sum_{m=0}^{\infty} F_{m(r)}a^{\dagger}_{m(r)}\,. \tag{5.2}$$

The Fourier transform of (5.1) leads to the equations [9]

$$\sum_{r=1}^{3}\left[X^{(r)}C^{1/2}_{(r)}F_{(r)}\right]_m = 0 = \sum_{r=1}^{3}\alpha_r\left[X^{(r)}C^{-1/2}_{(r)}F_{(r)}\right]_m. \tag{5.3}$$

Here the components $m = 0$ and $m > 0$ decouple from each other. For $m = 0$ one finds [9]

$$F_{0(1)} = -\sqrt{\frac{2}{\alpha'}}\sqrt{\mu\alpha_1}\alpha_2\,, \qquad F_{0(2)} = -\sqrt{\frac{\alpha_1}{\alpha_2}}F_{0(1)}\,, \qquad F_{0(3)} = 0\,. \tag{5.4}$$

Then $\mathcal{K}_0$ can be written as

$$\mathcal{K}_0 = \mathbb{P} - i\frac{\mu\alpha}{\alpha'}\mathbb{R} \tag{5.5}$$

which has the correct flat space limit. So in contrast to a statement in [8] the term $\mu\mathbb{R}$ is included in the supergravity part of the prefactor and although in supergravity $\mathbb{R}|V\rangle = 0$ we have $\mathcal{K}_0^I\mathcal{K}_0^J|V\rangle = [\mathbb{P}^I\mathbb{P}^J + \frac{\mu\alpha}{\alpha'}\delta^{IJ}]|V\rangle$ so the inclusion of $\mathbb{R}$ does make a difference.[5] The overall normalization of $F_{0(1)}$ is not determined by (5.3) and could be any function $f(\mu)$ with $\lim\limits_{\mu\to 0} f(\mu) = 1$.[6] For $m > 0$ we have

$$\sum_{r=1}^{3}\left[A^{(r)}C^{-1/2}C_{(r)}^{1/2}F_{(r)}\right]_m = \frac{1}{\sqrt{\alpha'}}\mu\alpha B_m = \sum_{r=1}^{3}\mu\alpha_r\left[A^{(r)}C^{-1/2}C_{(r)}^{-1/2}F_{(r)}\right]. \tag{5.6}$$

Subtracting the second equation from the first one

$$\sum_{r=1}^{3}\left[A^{(r)}C^{1/2}C_{(r)}^{-1/2}U_{(r)}F_{(r)}\right]_m = 0\,. \tag{5.7}$$

Using the first identity in (A.5) gives [9]

$$F_{m(r)} = \frac{1}{\alpha_r}\left[C_{(r)}^{1/2}C^{1/2}U_{(r)}^{-1}A^{(r)\,T}V\right]_m \tag{5.8}$$

with $V_m$ an arbitrary vector determined by plugging the above expression for $F_{m(r)}$ in, say, the second equation in (5.6). The complete solution is [9]

$$F_{m(r)} = \frac{\alpha}{\sqrt{\alpha'}}\frac{1}{\alpha_r}\left[C_{(r)}^{1/2}C^{1/2}U_{(r)}^{-1}A^{(r)\,T}\Upsilon^{-1}B\right]_m\,. \tag{5.9}$$

The matrix $\Upsilon$ was introduced in section 4. Using eq. (4.7) one can show that

$$F_{m(r)} = -\frac{1}{\sqrt{\alpha'}}\frac{\alpha}{1-4\mu\alpha K}\frac{1}{\alpha_r}\left[U_{(r)}^{-1}C_{(r)}^{1/2}C\overline{N}^r\right]_m \tag{5.10}$$

and as $\mu \to 0$

$$\lim_{\mu\to 0}(\mathcal{K}_0 + \mathcal{K}_+) = \mathbb{P} - \frac{\alpha}{\sqrt{\alpha'}}\sum_{r=1}^{3}\sum_{m=1}^{\infty}\frac{1}{\alpha_r}\left[C\overline{N}^r\right]_m\sqrt{m}a_{m(r)}^\dagger \tag{5.11}$$

coincides with the flat space result of [7].

[5]Moreover $\mathbb{R}|V\rangle$ is no longer zero in the full string theory, so to obtain a consistent expression for the prefactor $\mathbb{R}$ has to be included, as was in fact done in [9] when working with the oscillator expressions.

[6]In [15] this normalization was fixed to be a constant by comparing the results of [8] with a supergravity calculation.

From the expression for $F_{m(r)}$ in (5.10) we see that the equations in (5.3) are actually constraints on $\overline{N}^r$

$$B + \sum_{r=1}^{3} A^{(r)} C^{1/2} U_{(r)} \overline{N}^r = 0\,, \qquad \sum_{r=1}^{3} \frac{1}{\alpha_r} A^{(r)} C^{3/2} \overline{N}^r = 0 \tag{5.12}$$

precisely satisfied by $\overline{N}^r = -C^{-1/2} A^{(r)\,T} \Gamma^{-1} B$.

Now take into account the negative modes, i.e. consider

$$\mathcal{K}_- = \sum_{r=1}^{3} \sum_{m=1}^{\infty} F_{-m(r)} a^\dagger_{-m(r)}\,. \tag{5.13}$$

This leads to the equations [9]

$$\sum_{r=1}^{3} \frac{1}{\alpha_r} \left[ A^{(r)} C^{1/2} C^{1/2}_{(r)} F_{(r)} \right]_{-m} = 0 = \sum_{r=1}^{3} \left[ A^{(r)} C^{1/2} C^{-1/2}_{(r)} F_{(r)} \right]_{-m}\,. \tag{5.14}$$

Comparing the second equation with (5.7) it is clear that

$$F_{-m(r)} \sim U_{m(r)} F_{m(r)}\,. \tag{5.15}$$

However, if one substitutes this into the first equation one actually sees that the sum is divergent [6, 7, 9]. This phenomenon already appears in flat space and it is known [6] that the function of $\sigma$ responsible for the divergence is $\delta(\sigma - \pi\alpha_1) - \delta(\sigma + \pi\alpha_1)$. Since $\pm\pi\alpha_1$ are actually identified this divergence is harmless but the precise normalization in (5.15) has to be determined by other means.

In flat space the prefactor can be alternatively defined as follows. Consider the operators defined via their action on $|V\rangle$ [6, 7]

$$\begin{aligned} \partial X |V\rangle &= 4\pi \frac{\sqrt{-\alpha}}{\alpha'} \lim_{\sigma\to\pi\alpha_1} (\pi\alpha_1 - \sigma)^{1/2} \left( \dot{x}_1(\sigma) + \dot{x}_1(-\sigma) \right) |V\rangle\,, \\ P|V\rangle &= -2\pi\sqrt{-\alpha} \lim_{\sigma\to\pi\alpha_1} (\pi\alpha_1 - \sigma)^{1/2} \left( p_1(\sigma) + p_1(-\sigma) \right) |V\rangle\,. \end{aligned} \tag{5.16}$$

These expressions contain creation and annihilation operators. The claim is that after commuting the annihilation operators through the exponential and taking the limit $\sigma \to \pi\alpha_1$ we have[7]

$$\begin{aligned} \left( P + \frac{1}{4\pi} \partial X \right) |V\rangle &= \left( \mathcal{K}_0 + \mathcal{K}_+ + \mathcal{K}_- \right) |V\rangle \equiv \mathcal{K} |V\rangle\,, \\ \left( P - \frac{1}{4\pi} \partial X \right) |V\rangle &= \left( \mathcal{K}_0 + \mathcal{K}_+ - \mathcal{K}_- \right) |V\rangle \equiv \widetilde{\mathcal{K}} |V\rangle\,. \end{aligned} \tag{5.17}$$

Substituting the mode expansion of $p_1(\sigma)$ into (5.16) yields

$$\begin{aligned} P|V\rangle &\equiv \sum_{r=1}^{3} \sum_{m=0}^{\infty} P_{m(r)} a^\dagger_{m(r)} |V\rangle = \\ &= -\frac{2}{\alpha_1} \frac{\sqrt{-\alpha}}{\sqrt{\alpha'}} \lim_{e\to 0} \varepsilon^{1/2} \sum_{r=1}^{3} \sum_{m=0}^{\infty} \left[ \sum_{n=1}^{\infty} (-1)^n \sqrt{\omega_{n(1)}} \cos\left( \frac{n\varepsilon}{\alpha_1} \right) \overline{N}^{1r}_{nm} \right] a^\dagger_{m(r)} |V\rangle\,. \end{aligned} \tag{5.18}$$

[7]Notice that in comparison with [8, 9] we changed the sign in the definition of $\partial X$. Then $\mathcal{K} \leftrightarrow X_{\rm GS}$ and $\widetilde{\mathcal{K}} \leftrightarrow \widetilde{X}_{\rm GS}$ as compared to [6].



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Now the singular behavior of the sum as $\varepsilon \to 0$ can be traced to the way it diverges as $n \to \infty$. Therefore to take the limit $\varepsilon \to 0$ we can approximate the summand for large $n$. Thus for $m > 0$ and $n$ large

$$\begin{aligned} \overline{N}^{1r}_{n0} &\sim -\sqrt{2\mu\alpha_r n}\varepsilon^{rt}\alpha_t \overline{N}^1_n\,, \\ \overline{N}^{1r}_{nm} &\sim -\frac{\alpha}{\alpha_r}\frac{1}{1-4\mu\alpha K}\left(C^{1/2}\overline{N}^1\right)_n \left(U^{-1}_{(r)}C^{1/2}_{(r)}C\overline{N}^r\right)_m\,, \end{aligned} \tag{5.19}$$

where we used (4.11). Hence we find for $m > 0$

$$P_{0(1)} = f(\mu)F_{0(1)}\,, \qquad P_{0(2)} = -\sqrt{\frac{\alpha_1}{\alpha_2}}P_{0(1)}\,, \qquad P_{m(r)} = f(\mu)F_{m(r)}\,. \tag{5.20}$$

Here we defined

$$f(\mu) \equiv -2\frac{\sqrt{-\alpha}}{\alpha_1}\lim_{e\to 0}\varepsilon^{1/2}\sum_{n=1}^{\infty}(-1)^n n\cos\left(\frac{n\varepsilon}{\alpha_1}\right)\overline{N}^1_n \tag{5.21}$$

which is equal to one for $\mu = 0$ [6]. It is not clear to us that or even if this still holds for $\mu$ non zero but we will not need the precise form of $f(\mu)$ in what follows. Thus we have shown that $P|V\rangle = f(\mu)\left(\mathcal{K}_0 + \mathcal{K}_+\right)|V\rangle$. For the negative modes we get

$$\begin{aligned} \frac{1}{4\pi}\partial X|V\rangle &\equiv \sum_{r=1}^{3}\sum_{m=1}^{\infty}\partial X_{-m(r)}a^{\dagger}_{-m(r)}|V\rangle = \\ &= \frac{2i}{\sqrt{\alpha'}}\frac{\sqrt{-\alpha}}{\alpha_1}\lim_{\varepsilon\to 0}\varepsilon^{1/2}\sum_{r=1}^{3}\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{\omega_{n(1)}}}\cos(n\varepsilon/\alpha_1)\overline{N}^{rs}_{-n,-m}\right]a^{\dagger}_{-m(r)}|V\rangle\,. \end{aligned} \tag{5.22}$$

Using that $\overline{N}^{1r}_{-n,-m} \sim -\overline{N}^{1r}_{nm}U_{m(r)}$ for large $n$ we see that

$$\partial X_{-m(r)} = iU_{m(r)}P_{m(r)}\,. \tag{5.23}$$

Hence $F_{-m(r)} = iU_{m(r)}F_{m(r)}$ which differs by a factor of $i$ from the result of [9]. As $\mu \to 0$ it reduces to the flat space result of [7]. So we have seen that the continuum basis operators and the oscillator expressions coincide up to a possibly $\mu$-dependent normalization. Since we do not fix the normalization anyway this is not important. We will see however in the next subsection that the situation is slightly different in the fermionic case.

### 5.2 The fermionic constituent

The fermionic part of the prefactor has to satisfy the conditions

$$\left\{\sum_{r=1}^{3}\lambda_r(\sigma), \widetilde{G}\right\} = 0 = \left\{\sum_{r=1}^{3}e(\alpha_r)\vartheta_r(\sigma), \widetilde{G}\right\}\,. \tag{5.24}$$

Consider

$$\mathcal{Y} = \sum_{r=1}^{2}G_{0(r)}\lambda_{0(r)} + \sum_{r=1}^{3}\sum_{m=1}^{\infty}G_{m(r)}b^{\dagger}_{m(r)}\,. \tag{5.25}$$

For the zero-modes we can set the coefficient of, say, $\lambda_{0(3)}$ to zero due to the property of the fermionic supergravity vertex that $\sum_{r=1}^{3}\lambda_{0(r)}|E_b^0\rangle = 0$ . The Fourier transform of (5.24) leads to the equations

$$\sum_{r=1}^{3}\frac{1}{\sqrt{|\alpha_r|}}\left[A^{(r)}CC_{(r)}^{-1/2}P_{(r)}G_{(r)}\right]_m = 0\,,$$

$$-\left[C^{1/2}B\right]_m\sum_{r,s=1}^{2}\varepsilon^{rs}\alpha_r\alpha_s G_{0(r)} + \sum_{r=1}^{3}e(\alpha_r)\sqrt{|\alpha_r|}\left[C^{1/2}A^{(r)}C_{(r)}^{-1/2}P_{(r)}^{-1}G_{(r)}\right]_m = 0\,. \quad (5.26)$$

The components $m = 0$ and $m > 0$ decouple from each other (the term proportional to $B$ is absent for $m = 0$). For $m = 0$ we have

$$G_{0(1)} = -\sqrt{\frac{2}{\alpha'}}\alpha_2\,, \qquad G_{0(2)} = \sqrt{\frac{2}{\alpha'}}\alpha_1\,. \quad (5.27)$$

As in the previous subsection the normalization is only determined up to a matrix $y_{ab}(\mu)$ with $\lim_{\mu\to 0} y_{ab}(\mu) = \delta_{ab}$. For $m > 0$ we can rewrite the second equation as

$$\sum_{r=1}^{3}e(\alpha_r)\sqrt{|\alpha_r|}\left[A^{(r)}C_{(r)}^{-1/2}P_{(r)}^{-1}G_{(r)}\right]_m = \frac{\alpha}{\sqrt{\alpha'}}B_m\,. \quad (5.28)$$

Then

$$G_{m(r)} = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\left[P_{(r)}^{-1}C_{(r)}^{1/2}A^{(r)\,T}W\right]_m \quad (5.29)$$

solves the first equation using (A.5) with $W_m$ an arbitrary vector that is determined by the second equation. The final solution is

$$G_{m(r)} = \frac{\alpha}{\sqrt{a'}}\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\left[P_{(r)}^{-1}C_{(r)}^{1/2}A^{(r)\,T}\widetilde{\Upsilon}^{-1}B\right]_m\,. \quad (5.30)$$

Here

$$\widetilde{\Upsilon} \equiv \sum_{r=1}^{3}A^{(r)}P_{(r)}^{-2}A^{(r)\,T}\,, \qquad \widetilde{\Upsilon}^{-1} = \Gamma^{-1} - \frac{1}{2}\frac{\mu\alpha}{1-4\mu\alpha K}\left(\Gamma^{-1}B\right)\left(\Gamma^{-1}B\right)^T(1+\Pi)\,. \quad (5.31)$$

Hence $G_{(r)}$ can be expressed via $F_{(r)}$ as

$$G_{(r)} = \left(1-2\mu\alpha K(1-\Pi)\right)\sqrt{|\alpha_r|}P_{(r)}^{-1}U_{(r)}C^{-1/2}F_{(r)}\,. \quad (5.32)$$

As $\mu\to 0$ we have

$$\lim_{\mu\to 0}\mathcal{Y} = \sqrt{\frac{2}{\alpha'}}\Lambda + \sum_{r=1}^{3}\sum_{m=1}^{\infty}\frac{F_{m(r)}}{\sqrt{m}}\sqrt{|\alpha_r|}b_{m(r)}^{\dagger}\,. \quad (5.33)$$

Taking into account that $\sqrt{|\alpha_r|}b_{m(r)}^{\dagger} \leftrightarrow Q^I_{-m(r)}$ this is exactly the flat space expression of [7]. As a further check of the previous equations, in particular 5.32, we note that upon substitution of $G_{(r)}$ one can show that the constraints determining $G_{(r)}$ reduce again to the two equations given in (5.12). In flat space the operator

$$Y(\sigma) = -2\pi \frac{\sqrt{-2\alpha}}{\sqrt{\alpha'}} (\pi\alpha_1 - \sigma)^{1/2} \left(\lambda_1(\sigma) + \lambda_1(-\sigma)\right) \tag{5.34}$$

satisfies $\lim_{\sigma\to\pi\alpha_1} Y(\sigma)|V\rangle = \mathcal{Y}|V\rangle$. Substituting the mode expansion for $\lambda_1(\sigma)$ we get

$$\begin{aligned} Y|V\rangle &= -\sqrt{\frac{2}{\alpha'}}\sqrt{\frac{-2\alpha}{\alpha_1}} \lim_{\varepsilon\to 0} \varepsilon^{1/2} \sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{n\varepsilon}{\alpha_1}\right) \times \\ &\quad \times \left[\sqrt{2}\Lambda Q_n^1 + \sum_{r=1}^{3}\sum_{m=1}^{\infty} Q_{nm}^{1r} b_{m(r)}^{\dagger}\right] |V\rangle \,. \end{aligned} \tag{5.35}$$

For large $n$

$$\begin{aligned} Q_n^1 &\sim \frac{1}{\sqrt{\alpha_1}}(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\left[C\overline{N}^1\right]_n, \\ Q_{nm}^{1r} &\sim \sqrt{\frac{\alpha'}{\alpha_1}}(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\left[C\overline{N}^1\right]_n G_{m(r)} \,. \end{aligned} \tag{5.36}$$

Then

$$\begin{aligned} Y_0 &= f(\mu)(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\sqrt{\frac{2}{\alpha'}}\Lambda \,, \\ Y_{m(r)} &= f(\mu)(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi)) G_{m(r)} \,. \end{aligned} \tag{5.37}$$

So we have shown that $Y|V\rangle = f(\mu)(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\mathcal{Y}|V\rangle$, i.e. for the fermionic constituent a non-trivial $\mu$-dependent matrix appears in the relative normalization.

## 6. Conclusions

In this paper we studied light-cone superstring field theory on the maximally supersymmetric pp-wave background. Our main results are a modified expression for the fermionic contribution to the exponential part of the vertex as compared to [8], the generalization of a flat space factorization theorem [6] to the pp-wave background (see also [22]) and simple expressions for the fermionic Neumann matrices in terms of the bosonic ones. We analyzed the oscillator and continuum basis expressions for the constituents of the prefactor in detail and pointed out that in contrast to flat space the relation between the two is non trivial in the fermionic case. As a consistency check, we have shown that all expressions appearing in the exponential as well as the prefactor part of the interaction vertex coincide with the flat space results of [7] in the limit $\mu \to 0$.

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## A. Some summation formulae

It is convenient to introduce the matrices for $m, n > 0$

$$\begin{aligned}
C_{mn} &= m\delta_{mn}\,, \\
A^{(1)}_{mn} &= (-1)^n \frac{2\sqrt{mn}\beta}{\pi} \frac{\sin m\pi\beta}{m^2\beta^2 - n^2} = \left(C^{-1/2} X^{(1)} C^{1/2}\right)_{mn}\,, \\
A^{(2)}_{mn} &= \frac{2\sqrt{mn}(\beta+1)}{\pi} \frac{\sin m\pi\beta}{m^2(\beta+1)^2 - n^2} = \left(C^{-1/2} X^{(2)} C^{1/2}\right)_{mn}\,, \\
A^{(3)}_{mn} &= \delta_{mn}
\end{aligned} \tag{A.1}$$

and the vector $(m > 0)$

$$B_m = -\frac{2}{\pi} \frac{\alpha_3}{\alpha_1\alpha_2} m^{-3/2} \sin m\pi\beta \tag{A.2}$$

related to $X^{(r)}_{m0}$ via

$$X^{(r)}_{m0} = -\varepsilon^{rs}\alpha_s \left(C^{1/2}B\right)_m\,. \tag{A.3}$$

These quantities satisfy some very important identities that were derived in [6]. They are

$$\begin{aligned}
-\frac{\alpha_3}{\alpha_r} C A^{(r)T} C^{-1} A^{(s)} &= \delta^{rs}\mathbf{1}\,, \\
-\frac{\alpha_r}{\alpha_3} C^{-1} A^{(r)T} C A^{(s)} &= \delta^{rs}\mathbf{1}\,, \qquad A^{(r)T} C B = 0
\end{aligned} \tag{A.4}$$

for $r, s \in \{1, 2\}$ and

$$\sum_{r=1}^{3} \frac{1}{\alpha_r} A^{(r)} C A^{(r)\,T} = 0\,, \qquad \sum_{r=1}^{3} \alpha_r A^{(r)} C^{-1} A^{(r)\,T} = \frac{\alpha}{2} B B^T\,, \tag{A.5}$$

where $\alpha \equiv \alpha_1\alpha_2\alpha_3$. In terms of the big matrices $X^{(r)}_{mn}$, $m, n \in \mathbb{Z}$ the relations (A.4) and (A.5) can be written in the compact form

$$\left(X^{(r)T} X^{(s)}\right)_{mn} = -\frac{\alpha_3}{\alpha_r}\delta^{rs}\delta_{mn}\,, \qquad r, s \in \{1, 2\}\,, \qquad \sum_{r=1}^{3} \alpha_r \left(X^{(r)} X^{(r)T}\right)_{mn} = 0\,. \tag{A.6}$$

## B. The exponential part of the vertex

### The bosonic part

The bosonic constraints the exponential part of the vertex has to satisfy are

$$\sum_{r=1}^{3} \sum_{n\in\mathbb{Z}} X^{(r)}_{mn} p_{n(r)} |V\rangle = 0\,, \qquad \sum_{r=1}^{3} \sum_{n\in\mathbb{Z}} \alpha_r X^{(r)}_{mn} x_{n(r)} |V\rangle = 0\,. \tag{B.1}$$

For $m = 0$ this leads to

$$\sum_{r=1}^{3} p_{0(r)} |V\rangle = 0\,, \qquad \sum_{r=1}^{3} \alpha_r x_{0(r)} |V\rangle = 0\,. \tag{B.2}$$

Substituting (2.3) and commuting the annihilation operators through the exponential this requires

$$\sum_{r,s=1}^{3} \sqrt{|\alpha_r|} \left[ \left(\overline{N}^{rs}_{00} + \delta^{rs}\right) a^{\dagger}_{0(s)} + \sum_{n=1}^{\infty} \overline{N}^{rs}_{0n} a^{\dagger}_{n(s)} \right] |V\rangle = 0\,, \tag{B.3}$$

$$\sum_{r,s=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|} \left[ \left(\overline{N}^{rs}_{00} - \delta^{rs}\right) a^{\dagger}_{0(s)} + \sum_{n=1}^{\infty} \overline{N}^{rs}_{0n} a^{\dagger}_{n(s)} \right] |V\rangle = 0\,. \tag{B.4}$$

Using the expressions given for $\overline{N}^{rs}_{0n}$ and $\overline{N}^{rs}_{00}$ in (4.2) and (4.3) and (4.4) one can check that the above equations are satisfied trivially, i.e. without further use of additional non-trivial identities. For $m > 0$ we find the following constraints

$$B + \sum_{r=1}^{3} A^{(r)} C^{1/2} U_{(r)} \overline{N}^{r} = 0\,, \tag{B.5}$$

$$A^{(s)} C_{(s)}^{-1/2} U_{(s)}^{-1} + \sum_{r=1}^{3} A^{(r)} C_{(r)}^{-1/2} U_{(r)} C^{1/2} \overline{N}^{rs} C^{-1/2} = 0\,, \tag{B.6}$$

$$-\alpha_s A^{(s)} C_{(s)}^{-1/2} + \sum_{r=1}^{3} \alpha_r A^{(r)} C_{(r)}^{-1/2} C^{-1/2} \overline{N}^{rs} C^{1/2} = \alpha B \left[ C_{(s)}^{1/2} C^{1/2} \overline{N}^{s} \right]^{T}. \tag{B.7}$$

Equation (B.5) is identical to the first equation in (5.12). To prove eqs. (B.6) and (B.7) substitute the expression for $\overline{N}^{rs}$ given in (4.1). For $m < 0$ there is one additional constraint

$$A^{(s)} C_{(s)}^{-1/2} U_{(s)}^{-1} - \alpha_s \sum_{r=1}^{3} \frac{1}{\alpha_r} A^{(r)} C_{(r)}^{1/2} U_{(r)} C^{1/2} \overline{N}^{rs} C^{-1/2} C_{(s)}^{-1} = 0 \tag{B.8}$$

which can be verified by subtracting it from eq. (B.6) and using (4.12). Here the identity

$$\sum_{r=1}^{3} \alpha_r A^{(r)} C^{-1/2} \overline{N}^{r} = 2\alpha K B\,. \tag{B.9}$$

is used.

### The fermionic part

The fermionic constraints the exponential part of the vertex has to satisfy are

$$\sum_{r=1}^{3} \sum_{n\in\mathbb{Z}} X^{(r)}_{mn} \lambda_{n(r)} |V\rangle = 0\,, \qquad \sum_{r=1}^{3} \sum_{n\in\mathbb{Z}} \alpha_r X^{(r)}_{mn} \vartheta_{n(r)} |V\rangle = 0\,. \tag{B.10}$$

For $m = 0$ this leads to

$$\sum_{r=1}^{3} \lambda_{0(r)} |V\rangle = 0\,, \qquad \sum_{r=1}^{3} \alpha_r \vartheta_{0(r)} |V\rangle = 0\,. \tag{B.11}$$

These equations are satisfied by construction of the zero-mode part of $|V\rangle$. For $m > 0$ we get

$$B + \sum_{r=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}A^{(r)}C_{(r)}^{-1/2}P_{(r)}Q^r = 0\,, \tag{B.12}$$

$$\sqrt{|\alpha_s|}A^{(s)}C_{(s)}^{-1/2}P_{(s)}^{-1} + \sum_{r=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}A^{(r)}C_{(r)}^{-1/2}P_{(r)}Q^{rs} = 0\,, \tag{B.13}$$

$$-\sqrt{|\alpha_s|}A^{(s)}C_{(s)}^{-1/2}P_{(s)} + \sum_{r=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}A^{(r)}C_{(r)}^{-1/2}P_{(r)}^{-1}Q^{sr\,T} = \alpha B Q^{s\,T} \tag{B.14}$$

and $\alpha_3\Theta|V_0\rangle \equiv (\vartheta_{0(1)} - \vartheta_{0(2)})|V_0\rangle = 0$. The latter equation is satisfied by the zero-mode part of the vertex given in eq. (3.18). For $m < 0$ the constraints are

$$\sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}}A^{(r)}CC_{(r)}^{-1/2}P_{(r)}^{-1}Q^r = 0\,, \tag{B.15}$$

$$A^{(s)}CC_{(s)}^{-1/2} - e(\alpha_s)\sqrt{|\alpha_s|}\sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}}A^{(r)}CC_{(r)}^{-1/2}P_{(r)}^{-1}Q^{rs}P_{(s)}^{-1} = 0\,, \tag{B.16}$$

$$A^{(s)}CC_{(s)}^{-1/2} + e(\alpha_s)\sqrt{|\alpha_s|}\sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}}A^{(r)}CC_{(r)}^{-1/2}P_{(r)}Q^{sr\,T}P_{(s)} = 0\,. \tag{B.17}$$

Now eqs. (B.12) and (B.15) uniquely determine

$$Q^r = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}(1 - 4\mu\alpha K)^{-1}(1 - 2\mu\alpha K(1+\Pi))P_{(r)}C_{(r)}^{1/2}C^{1/2}\overline{N}^r\,. \tag{B.18}$$

Furthermore comparing eqs. (B.13) and (B.6) we see that

$$Q^{rs} = e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}P_{(r)}^{-1}U_{(r)}C^{1/2}\overline{N}^{rs}C^{-1/2}U_{(s)}P_{(s)}^{-1} \tag{B.19}$$

solves (B.13). Using

$$P_{(r)}^{-2}U_{(r)}\overline{N}^{rs}U_{(s)}P_{(s)}^{-2} = \overline{N}^{rs} + \mu\alpha(1 - 4\mu\alpha K)^{-1}C_{(r)}^{1/2}\overline{N}^r\left[C_{(s)}^{1/2}\overline{N}^s\right]^T(1-\Pi) \tag{B.20}$$

establishes (B.14) by virtue of (B.7). Equation (B.16) is satisfied due to the identity

$$A^{(s)}C_{(s)}^{-1/2} - \alpha_s\sum_{r=1}^{3}\frac{1}{\alpha_r}A^{(r)}C_{(r)}^{-1/2}C^{3/2}\overline{N}^{rs}C^{-3/2} = 0 \tag{B.21}$$

which can be proved using the expression for $\overline{N}^{rs}$ given in (4.1). Finally, eq. (B.17) is identical to (B.6). This concludes the determination of the exponential part of the fermionic vertex.

## C. A note on the fermionic zero-mode vertex

Here we prove that the recent proposal of [20] for the fermionic zero-mode vertex leads to inconsistencies not visible in supergravity, if one requires the vertex to have an exponential form. Recall that their proposal is to use [20]

$$|\delta\rangle = \prod_{a=1}^{8} \left[ \sum_{r,s=1}^{3} \alpha_s \lambda^a_{0(r)} \vartheta^a_{0(s)} \right] |v\rangle \tag{C.1}$$

instead of $|E_b^0\rangle$ as the fermionic zero-mode vertex. Here $|v\rangle$ is annihilated by all the $b_{0(r)}$ and breaks SO(8) to the $\mathrm{SO}(4)\times\mathrm{SO}(4)\times\mathbb{Z}_2$ symmetry of the pp-wave background. Expressing $\lambda_{0(r)}$ and $\vartheta_{0(r)}$ in terms of $b_{0(r)}$, $b^\dagger_{0(r)}$ it is easy to see that

$$|\delta\rangle = \alpha_3^8 \prod_{a=1}^{8} \left[ 1 - \left( \sqrt{-\beta}\, b^\dagger_{0(1)} + \sqrt{\beta+1}\, b^\dagger_{0(2)} \right) b^\dagger_{0(3)} \right] |v\rangle \,. \tag{C.2}$$

Let us introduce

$$\mathcal{Y}_0 \equiv \sqrt{\frac{\alpha_1\alpha_2}{2}} \left[ \sqrt{\alpha_1}\, b^\dagger_{0(2)} - \sqrt{\alpha_2}\, b^\dagger_{0(1)} \right] . \tag{C.3}$$

Then one can show

$$\Lambda|\delta\rangle = \frac{1}{2}(1+\Pi)\mathcal{Y}_0|\delta\rangle \,, \qquad \Theta|\delta\rangle = -\frac{1}{\alpha}(1-\Pi)\mathcal{Y}_0|\delta\rangle \,, \tag{C.4}$$

i.e. $(1-\Pi)\Lambda|\delta\rangle = 0 = (1+\Pi)\Theta|\delta\rangle$. Now the most general ansatz for the exponential part of the fermionic vertex is

$$\begin{aligned} |E_b\rangle \sim \exp \Bigg[ & \sum_{r,s=1}^{3} \sum_{n=1}^{\infty} b^\dagger_{0(r)} \left( Q^{rs}_{0n} b^\dagger_{-n(s)} + \widetilde{Q}^{rs}_{0n} b^\dagger_{n(s)} \right) + \\ & + \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} b^\dagger_{-m(r)} Q^{rs}_{mn} b^\dagger_{n(s)} \Bigg] |\delta\rangle \,. \end{aligned} \tag{C.5}$$

Consider the first constraint in eq. (B.10). The $m=0$ component is satisfied by construction of $|\delta\rangle$ [20]. One can show that the conditions arising for $m>0$ and $m<0$ imply that

$$Q^{rs}_{0n} = \frac{1}{2}(1+\Pi)\varepsilon^{rt}\sqrt{\alpha_r}\,\alpha_t Q^s_n \,, \qquad \widetilde{Q}^{rs}_{0n} = 0 \tag{C.6}$$

and furthermore reduce to the eqs. (B.12), (B.13) and (B.17) of appendix B. In fact, we could have started with the expressions for $Q^{rs}$ and $\widetilde{Q}^{rs}$ right away in the ansatz for $|E_b\rangle$, because we know that $\Lambda$ will appear in $|E_b\rangle$ in the way it does in eq. (3.17) since this term arises from integration over the non-zero-modes, which is independent of the choice for the zero-mode vacuum. Eliminating the part in $\Lambda$ which contains annihilation operators (cf. eq. (C.4)) then yields eq. (C.6). Now the $m=0$ component of the constraint involving

$\vartheta_r(\sigma_r)$ is satisfied by construction of $|\delta\rangle$ [20] and the $m<0$ component results in eqs. (B.15) and (B.16). However the $m>0$ component requires that

$$-\frac{1}{\alpha}(1-\Pi)\mathcal{Y}_0|\delta\rangle = 0\,,$$
$$-\sqrt{|\alpha_s|}A^{(s)}C_{(s)}^{-1/2}P_{(s)} + \sum_{r=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}A^{(r)}C_{(r)}^{-1/2}P_{(r)}^{-1}Q^{sr\,T} = \frac{1}{2}(1+\Pi)\alpha BQ^{s\,T}\,. \qquad \text{(C.7)}$$

Both of these equations are not consistent. Clearly the first equation (which is equivalent to the condition $\Theta|\delta\rangle = 0$, cf. eq. (C.4) is not satisfied, since $\mathcal{Y}_0$ just contains creation operators. The second equation is the same as (B.14), except for the factor of $\frac{1}{2}(1+\Pi)$ on the r.h.s. But we know that $Q^r$ and $Q^{rs}$ are uniquely determined by the eqs. (B.12)–(B.17) so this leads to a contradiction as well, since all the other equations are unchanged. Hence, with an exponential form of the vertex, $|\delta\rangle$ is not a consistent choice for its fermionic zero-mode part.

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